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Nonlinear stability bounds for inviscid, two-dimensional, parallel or circular flows with monotonic vorticity, and the analogous three-dimensional quasi-geostrophic flows

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Rigorous bounds are obtained on the mean normal displacement of vorticity or potential vorticity contours from their undisturbed parallel (or concentric) positions for incompressible planar flow, flow on the surface of a sphere, and three-dimensional quasi-geostrophic flow. It is required that the basic flows have monotonic distributions of vorticity, and it is this requirement that turns a particular linear combination of conserved quantities, a combination involving the linear or angular impulse and the areas enclosed by vorticity contours, into a norm when viewed in a certain hybrid Eulerian–Lagrangian set of coordinates. Liapunov stability theorems constraining the growth of finite-amplitude disturbances then follow merely from conservation of this norm. As a corollary, it is proved that arbitrarily steep, one-signed vorticity gradients are stable, including the limiting case of a circular patch of uniform vorticity.

1. Introduction

Recent advances in nonlinear stability theory following Arnol'd (1966) have provided methods for bounding the finite-amplitude growth of disturbances to large classes of fluid equilibria (e.g. Holm *et al.* 1985; Abarbanel *et al.* 1986; McIntyre & Shepherd 1987 and references therein). This paper presents a new approach to constructing the norms used in the theory, leading to new results on the stability of steady, two-dimensional, vortical flows. For example, one problem which has hitherto eluded a complete solution is that of proving the Liapunov stability of a circular vortex patch in an unbounded fluid. Progress was made by Wan & Pulvirenti (1985) who used elaborate methods to obtain a Liapunov stability result for the bounded case. In the unbounded case, however, only stability in a non-standard sense could be proved. The present approach proves a full Liapunov stability result, and more, by far simpler means.

It is a corollary of Arnol'd's first stability theorem (Arnol'd 1966) that two-dimensional, inviscid, incompressible, steady flows with one-signed vorticity gradients bounded away from zero and infinity are stable to finite-amplitude disturbances. For parallel or circular basic flows, i.e. flows which possess translational or rotational symmetry, the norm bounding the growth of disturbances can be written entirely in terms of the disturbance enstrophy (Arnol'd 1966; McIntyre & Shepherd 1987, §6). In the case of parallel flows, for example, the disturbance enstrophy (the domain integral of the squared vorticity disturbance) is bounded by its initial value times the basic-state, vorticity-gradient ratio $|Q_y|_{\max}/|Q_y|_{\min}$ (McIntyre & Shepherd 1987, equation (6.28)). An analogous theorem holds for zonal flows on the sphere (the coordinate y being replaced by the axial coordinate z – see equation (A 2) of Shepherd 1987) and for circularly symmetric flows on the plane (y being replaced by the radius r squared – see below). However, the bounds provided by these stability theorems are not sharp for basic flows containing a wide range of vorticity gradients; and the theorems are inapplicable for piecewise-constant vorticity distributions such as the circular vortex patch, for which $|Q_y|_{\max} = \infty$ and $|Q_y|_{\min} = 0$.

New methods, distinct from those originated by Arnol'd, were developed by Wan & Pulvirenti (1985) in an attack on the vortex-patch problem. However, a proof of Liapunov stability could be obtained only when the vortex is enclosed within a finite circular rigid boundary. The norm used was the L^1 area norm A_1 (that is to say the magnitude of the areal displacement of the vortex boundary, it being assumed that the disturbance does not alter the area of the vortex). Let A denote the area of the vortex, A_{dom} the finite area of the entire domain, Q the uniform vorticity within the vortex (the exterior being irrotational), and ΔJ the increase in the angular impulse (defined below) resulting from disturbing the vortex. Then Wan & Pulvirenti's stability theorem may be written

$$A_1^2 \leqslant \frac{8\pi\,\Delta J}{Q} \leqslant A_1(2A_{\text{dom}} - A_1). \tag{1}$$

The leftmost inequality can be obtained by minimizing ΔJ for a given A_1, and the configuration which achieves this is a circular vortex of area $A - \frac{1}{2}A_1$ surrounded by a circular band of area $\frac{1}{2}A_1$ and vorticity Q the inside edge of which is at the position of the undisturbed vortex boundary. The rightmost inequality results from maximizing ΔJ for a given A_1, and in this case the configuration is obtained by opening up a circular hole of irrotational fluid of area $\frac{1}{2}A_1$ at the centre of the domain and placing a detached band of equal area and vorticity Q adjacent to the outer edge of the domain. This is why Liapunov stability is lost when the domain is extended to infinity (although the left-hand inequality still, of course, gives some information about the stability of the circular vortex patch).

2. The new stability theorem

In two-dimensional, inviscid, incompressible flows, fluid particles retain their vorticity Q (in the geophysical context, read 'potential vorticity' for 'vorticity'). Material conservation of vorticity not only implies the conservation of linear and angular impulse, but it also implies the conservation of any function of Q – in general, an infinity of constraints. For instance the area enclosed within each constant-Q contour is invariant. For piecewise-constant distributions of vorticity, this translates into a finite number of constraints, namely the conservation of the area enclosed within each contour separating distinct uniform values of Q.

The basic flows under consideration are functions of a single coordinate, y. The essential step is to choose y appropriately. One chooses y such that the streamwise component of the conserved impulse, or angular impulse as the case may be, can be written in the form

$$J = \iint yQ(x, y, t)\,\mathrm{d}x\,\mathrm{d}y, \tag{2}$$

where t is time and x is the streamwise coordinate describing the appropriate translational or rotational symmetry operation (streamwise distance in the parallel case, azimuthal angle in the circular cases). Another way of characterizing the choice of y is to say that $\mathrm{d}x\,\mathrm{d}y$ is the area element. In parallel flow, y is spanwise distance. In circular planar flow, $y = \frac{1}{2}r^2$, where r is radial distance from the centre of rotation. And, in spherical zonal flow, $y = z$, the distance along the axis of rotation.

First consider the case where the vorticity distribution is piecewise constant. Then the expression (2) for J may be expressed in terms of contour integrals (via Stokes's theorem) as

$$J = \tfrac{1}{2}\sum_i \Delta Q_i \oint_{C_i} y_i^2\,\mathrm{d}x, \tag{3}$$

where the quantities $y_i(x,t)$, possibly multivalued functions of x, denote the disturbed contour positions, and the constants ΔQ_i denote the vorticity jumps across the contours C_i. For a specific contour, ΔQ_i equals the vorticity to the immediate left of the contour minus that to the immediate right, the convention being that each contour is traversed leaving the inside to the left, except when x and y are Cartesian coordinates in which case the opposite convention applies. Similarly, the conserved area within each contour may be written

$$A_i = \oint_{C_i} y_i\,\mathrm{d}x. \tag{4}$$

Of course, J and the A_i are conserved not only for the disturbed flow

$$y_i(x,t) = y_{ei} + \eta_i(x,t)$$

but also for the basic state y_{ei}, and, not surprisingly, any combination of these conserved quantities is likewise conserved. The particular combination

$$\mu = J - J_e - \sum_i \Delta Q_i\, y_{ei}(A_i - A_{ei})$$

$$= \tfrac{1}{2}\sum_i \Delta Q_i \oint_{C_i} \eta_i^2\,(x,t)\,\mathrm{d}x, \tag{5}$$

is precisely quadratic in the disturbance, and, therefore, when all of the ΔQ_i have the same sign, $\|\eta\| = |\mu|^{\frac{1}{2}}$ qualifies as a norm, and we have

$$\|\eta\|(t) = \|\eta\|(0). \tag{6}$$

This of course implies Liapunov stability, and indeed more. Not only does a sufficiently small initial disturbance imply boundedness for all time of the (same) norm, but the boundedness applies for an initial disturbance of any magnitude whatever.

The more general situation of a continuous vorticity distribution is straightforwardly obtained by taking the limit of an infinite number of contours; $\Delta Q_i \to \mathrm{d}Q$, $\eta_i(x,t) \to \eta(x,t;Q)$. The condition that the ΔQ_i all have the same sign becomes the requirement that the basic flow be monotonic. In this limit,

$$\|\eta\|^2 = \frac{1}{2}\int_{Q_{\min}}^{Q_{\max}} \mathrm{d}Q \oint \eta^2(x,t;Q)\,\mathrm{d}x. \tag{7}$$

3. The vortex patch

As a special case, consider the stability of a circular vortex patch. In equilibrium, the vorticity is taken to be uniform and non-zero for $r \leqslant 1$ and zero for $r > 1$. Then, the bound on the evolution of the perturbation $\eta(\theta, t) = \frac{1}{2}(r^2(\theta, t) - 1)$ is

$$\oint \eta^2(\theta, t)\,\mathrm{d}\theta = \oint \eta^2(\theta, 0)\,\mathrm{d}\theta, \tag{8}$$

where the possibility that η may be multivalued is implicit in the θ integration.

4. Quasi-geostrophic flow

With little extra effort, stability bounds can also be obtained for three-dimensional, baroclinic, quasi-geostrophic flow with surface temperature gradients at the lower and upper boundaries and shallow topography at the lower boundary. McIntyre & Shepherd (1987, Appendix B) give a full exposition of the stability of quasi-geostrophic flow and more, and the reader is referred to their paper for the basic equations and definitions. The bounds follow simply because fluid particles retain their quasi-geostrophic potential vorticity while moving incompressibly on approximately horizontal surfaces, and the surface potential temperature distributions may be regarded as extensions of the interior potential vorticity distribution (Hoskins, McIntyre & Robertson 1985, §5*b*).

First consider plane parallel flows $Q(y, z)$ which are monotonic in the 'meridional' direction y at each vertical level z ($z \propto \log$ (pressure)). Let $\theta_1(y)$ and $\theta_2(y)$ denote the potential temperature distributions at the lower ($z = z_1$) and upper ($z = z_2$) surfaces, $\theta_s(z)$ the reference potential temperature distribution, $\rho(z)$ the reference density stratification, $h_B(y)$ the height of shallow surface topography, f_0 the constant Coriolis parameter, and $\lambda(z) = \mathrm{d}\theta_s/\mathrm{d}z$. It is assumed that the quantities

$$\hat{\theta}(y) = \theta_1(y) + \lambda(z_1)h_B(y)$$

and $\theta_2(y)$ vary monotonically with y, $\hat{\theta}$ in the same sense as Q and θ_2 in the opposite sense. Then, a straightforward calculation shows that the following norm is invariant:

$$\left.\begin{aligned}
\|\eta\|^2 = \int \mathrm{d}z\,\rho(z) \int_{Q_{\min}(z)}^{Q_{\max}(z)} \mathrm{d}Q \oint \mathrm{d}x\,\eta^2(x, z, t; Q) + \frac{\rho(z_1)f_0}{\lambda(z_1)} \int_{\hat{\theta}_{\min}}^{\hat{\theta}_{\max}} \mathrm{d}\hat{\theta} \oint \mathrm{d}x\,\eta_1^2(x, t; \hat{\theta}) \\
+ \frac{\rho(z_2)f_0}{\lambda(z_2)} \int_{\theta_{2\min}}^{\theta_{2\max}} \mathrm{d}\theta \oint \mathrm{d}x\,\eta_2^2(x, t; \theta), \\
\eta(x, z, t; Q) = y(x, z, t; Q) - y_e(z; Q), \\
\eta_1(x, t; \theta) = y_1(x, t; \hat{\theta}) - y_{1e}(\hat{\theta}), \\
\eta_2(x, t; \theta) = y_2(x, t; \theta) - y_{2e}(\theta).
\end{aligned}\right\} \tag{9}$$

Note that this norm is essentially just the mass-weighted version of expression (7). For axisymmetric basic flows, (9) still applies with the substitution of $\frac{1}{2}r^2$ for y and azimuthal angle for x.

5. Discussion

By an appropriate linear combination of conserved quantities, an exactly conserved norm can be constructed when the basic flow possesses translational or rotational symmetry, the vorticity is monotonic, and the disturbance is viewed, in Lagrangian terms, as normal y-displacements of the equilibrium vorticity contours. The contributions include the impulse associated with the spatial symmetry of the basic flow and the area enclosed within each vorticity contour. The constancy of the norm expresses the fact that the mean-square contour displacements may never grow, and may never decay.

For a given disturbed flow, what restrictions are there on the choice of the basic flow? Is there an optimal choice, one for which the stability bounds are tightest? The class of possible basic flows is delimited by the necessary requirement that they contain a range of equilibrium vorticity values as great as the range of disturbed vorticity values, for then and only then can the concept of a disturbance to a vorticity contour make sense. Furthermore, there is an optimal choice of the basic flow which minimizes the disturbance norm $\|\eta\|$, even in the presence of parallel or circular boundaries. A straightforward variational calculation gives $\int \eta \, dx = 0$ for all vorticity contours Q, or, since this integral is independent of time,

$$\oint dx\{y(x,0;Q)-y_e(Q)\} = 0. \tag{10}$$

Hence, for domains periodic in x (or in the appropriate limit),

$$y_e(Q) = \oint dx \, y(x,0;Q) \bigg/ \oint dx, \tag{11}$$

defines the optimal basic flow in terms of the initial condition. Implicit in all the x integrations is the possibility of multivalued y. Thus, (11) is the prescription for constructing the basic state which minimizes the disturbance norm and pays proper attention to the boundary conditions. It is exactly the same prescription used to define 'available potential energy' for a disturbed density distribution above a flat surface (Lorenz 1955; also see Holliday & McIntyre 1981). Even potentially unstable initial conditions, e.g. a non-monotonic profile of Q, may nevertheless have a monotonic profile of $y_e(Q)$ when calculated from (11), as a result of the implicit multivaluedness of $y(x,0;Q)$. (In finite domains, $y_e(Q)$ is in fact guaranteed to be monotonic.) Suppose, for example, that $Q(x,y,0) = Q(y) +$ an infinitesimal disturbance, and $Q(y)$ has the S-shaped profile shown in figure 1. Then for the cross-section shown in the figure, $y_e(Q) = y_1 - y_2 + y_3$, and $\|\eta\|^2$, proportional to

$$\int dQ(y_1-y_2)(y_2-y_3),$$

is finite although the initial vorticity perturbation is infinitesimal. The finiteness of the norm expresses the fact that the initial contour perturbation is finite, and the invariance of the norm implies that the linear instability will therefore saturate at finite amplitude (cf. Shepherd 1988).

Wan & Pulvirenti (1985) also proved stability for perturbations to the circular vortex patch which introduce new vorticity values initially, but again Liapunov stability could only be obtained in the finite domain case. The preceding discussion in fact outlines the procedure for obtaining Liapunov stability in terms of contour

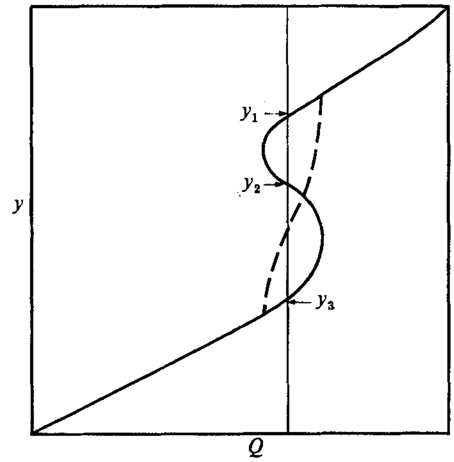

FIGURE 1. The basic flow ($y_e(Q)$, dashed line) corresponding to the disturbed initial condition ($Q(y)$, solid line) which minimizes the y-displacement norm while preserving vorticity measure.

displacements: simply rearrange the initial vorticity distribution by putting the fluid particles with the highest value of vorticity around the origin, the centre of rotation, and those with successively lower values at successively greater radii. The resultant circular distribution of vorticity is then the basic state which minimizes the vorticity-contour displacement norm $\|\eta\|$ whose invariance then proves Liapunov stability. The proof of stability fails, however, if the vorticity distribution contains both signs of vorticity in an infinite domain for which the fluid is irrotational at infinity, precisely because a monotonic basic state cannot be constructed from (11) without sending the fluid particles of one sign to infinity.

Finally, we note that the expression for $\|\eta\|^2$ in (7) can be regarded as a pseudomomentum (or angular pseudomomentum) in the sense discussed in McIntyre & Shepherd (1987, §7).

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